# CONSTRUCTION OF THE SOLUTIONS OF LINEAR OPERATOR EQUATIONS IN BANACH SPACES

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Formulas are obtained for the construction of the generalized inverse operator, resolving a linear Noetherian boundary value problem in a Banach space. The first of them is based on the construction of the generalized Green operator of the initial semi-homogeneous boundary value problem, while the second one is based on the application of certain results of the theory of linear operators in Banach spaces.

It is known [1, 2] that the boundary value problems for the system of functional-differential operators

$$\Lambda x = \begin{bmatrix} Lx \\ lx \end{bmatrix} = \begin{bmatrix} f \\ \alpha \end{bmatrix}$$

in the case when the dimension n of the functional-differential system Lx = f does not coincide with the dimension m of the vector functional $\ell$, are Noetherian. For such problems Schmidt's construction [3], which gives the possibility to construct the generalized inverse

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operator in the case of a Fredholm ($m = n$) boundary value problem, cannot be applied. We present two methods for the construction of the generalized inverse operator $\Lambda^-$ of a Noetherian operator $\Lambda$ in a Banach space. The first of them assumes the use of the properties of the operator L and is based on the construction of the generalized Green operator [2, 4] of the initial semi-homogeneous boundary value problem, while the second one is based on the application of certain results of the theory of operators in Banach spaces, enabling us to obtain an analogue of Schmidt's construction for Noetherian operators.

1. In the notations of [1, 5] we consider the boundary value problem

$$Lx = f, \tag{1}$$

$$lx = \alpha, \tag{2}$$

where $L: D_p^n \to L_p^n$ is a bounded linear operator for which the Cauchy problem $Lx = f$, $x(a) = c$ is uniquely solvable for any $f \in L_p^n$ and $c \in R^n$ and its solution has the form

$$x(t) = X(t)c + \int_a^b C(t, \tau) f(\tau)\, d\tau; \tag{3}$$

$X(t)$ is the $n \times n$ fundamental matrix of the operator $L$: $LX = 0$, $X(a) = E_n$, $C(t, \tau)$ is the $n \times n$ Cauchy matrix, which everywhere in the sequel will be considered defined in the square $[a, b] \times [a, b]$, setting $C(t, \tau) \equiv 0$ for $a \leqslant t < \tau \leqslant b$; $l: D_p^n \to R^m$ is a bounded linear vector-valued functional; $L_p^n$ is the space of n-dimensional vector-valued functions, having summable p-th powers, $1 < p < +\infty$, on the finite segment $[a, b]$; $D_p^n$ is the space of n-dimensional vector-valued functions, absolutely continuous on $[a, b]$ and such that $\dot{x} \in L_p^n$.

Such problems are, for example, the boundary value problems for ordinary differential equations [2], equations with delay [6], and problems with impulse actions [7].

Let $Q = lX$ be an $m \times n$ constant matrix; let $P_Q$ ($P_{Q^*}$) be the $n \times n$ ($m \times m$) orthoprojection, projecting $R^n$ ($R^m$) onto the null-space $N(Q)$ [$N(Q^*)$] of the matrix $Q$ ($Q^*$); let $P_{Q_r}$ be the $n \times r$ matrix ($r = n - \operatorname{rank} Q$) consisting of $r$ linearly independent columns of the matrix $P_Q$; let $P_{Q_d^*}$ be the $d \times m$ matrix ($d = m - \operatorname{rank} Q$) consisting of $d$ linearly independent rows of the matrix $P_{Q^*}$; let $Q^+$ be the unique Moore–Penrose generalized inverse $n \times m$ matrix, for the construction of which there exist detailed algorithms [8, 9].

<u>THEOREM 1.</u> If $\operatorname{rank} Q = n_1 < n$, then the boundary value problem (1), (2) is solvable for those and only those $f \in L_p^n$ and $\alpha \in R^m$ which satisfy the condition

$$P_{Q_d^*}\left\{\alpha - l\int_a^b C(\cdot, \tau) f(\tau)\, d\tau\right\} = 0, \tag{4}$$

and, moreover, we have an $r = n - n_1$-parameter family of solutions

$$x(t) = X_r(t) c_r + X(t) Q^+ \alpha + (Gf)(t), \tag{5}$$

where $X_r(t) = X(t)P_{Q_r}$ is the $n \times r$ fundamental matrix of the boundary value problem (1), (2) and G is the generalized Green operator of the semi-homogeneous boundary value problem (1), (2), defined in the following manner:

$$(Gf)(t) = \int_a^b C(t, \tau) f(\tau)\, d\tau - X(t) Q^+ l \int_a^b C(\cdot, \tau) f(\tau)\, d\tau. \tag{6}$$

<u>Proof.</u> The solution (3) of Eq. (1) is a solution of the boundary value problem (1), (2) if and only if the vector $c \in R^n$ satisfies the equation

$$Qc = \alpha - l\int_a^b C(\cdot, \tau) f(\tau)\, d\tau. \tag{7}$$

For the solvability of the algebraic system (7) and, consequently, also of the boundary value problem (1), (2), it is necessary and sufficient [8] that the right-hand side should belong to the orthogonal complement of the subspace $N(Q^*)$, i.e., the condition (4) should hold. In this case the algebraic system (7) has the solution

$$c = P_{Q_r} c_r + Q^+\left(\alpha - l\int_a^b C(\cdot, \tau) f(\tau)\, d\tau\right). \tag{8}$$

Introducing (8) into (3), we obtain the general solution (5) of the boundary value problem (1), (2).

The constructed generalized Green operator (6) solves the semi-homogeneous boundary value problem

$$\Lambda x = \begin{bmatrix} Lx \\ lx \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

and, as one can easily verify by a straightforward computation, satisfies the following relations:

$$\Lambda G * = \operatorname{col}\left[I*,\ P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau\right], \tag{9}$$

$$\lim_{t\to\tau+0} G* - \lim_{t\to\tau-0} G* = \int_a^b I * d\tau, \tag{10}$$

$$G*|_{t=a} = \int_a^b X P_{Q^*} * d\tau. \tag{11}$$

Making use of the properties (9)-(11) of the generalized Green operator of the boundary value problem (1), (2), one can show that the operator

$$\Lambda^- * = [G*,\ XQ^+ *] \tag{12}$$

is a generalized inverse [2] of the Noetherian operator $\Lambda$ and satisfies its defining [10, 11] properties:

$$\Lambda^-\Lambda\Lambda^- = \Lambda^-; \qquad \Lambda\Lambda^-\Lambda = \Lambda. \tag{13}$$

As shown in [10], the second property is a consequence of the first one. We verify the first property. Since $LX = 0$, $\ell X = Q$, making use of (9)-(11), we have

$$\Lambda\Lambda^- * = \begin{bmatrix} L \\ l \end{bmatrix} [G*,\ XQ^+ *] = \begin{bmatrix} LG* & LXQ^+ * \\ lG* & lXQ^+ * \end{bmatrix} = \begin{bmatrix} I_n * & 0 \\ P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau & QQ^+ * \end{bmatrix}.$$

Since $Q^+ P_{Q^*} = (Q^* Q + P_Q)^{-1} Q^* P_{Q^*} = 0$, $Q^+ QQ^+ = Q^+$ [9], we have

$$\Lambda^-\Lambda\Lambda^- * = [G,\ XQ^+] \begin{bmatrix} I_n * & 0 \\ P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau & QQ^+ * \end{bmatrix} = \left[G I_n * + XQ^+ P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau,\ XQ^+ QQ^+ *\right] = \Lambda^- *.$$

Thus, the operator $\Lambda^-$ is a generalized inverse of the Noetherian operator $\Lambda$. Making use of the form of $\Lambda$ and $\Lambda^-$, it is easy to verify that

$$P_{\Lambda^*} = I - \Lambda\Lambda^-,$$

where $P_{\Lambda^*}: L_p^n \times R^m \to \ker \Lambda^*$ is a projection, projecting the space $L_p^n \times R^m$ onto the kernel of the operator $\Lambda^*$. Indeed, since $I_m - QQ^+ = P_{Q^*}$ [9], we have

$$P_{\Lambda^*} * = \begin{bmatrix} I_n * & 0 \\ 0 & I_m * \end{bmatrix} - \begin{bmatrix} I_n * & 0 \\ P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau & QQ^+ * \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 \\ -P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau & (I_m - QQ^+) * \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau & P_{Q^*} * \end{bmatrix}.$$

Thus, the action of the projection $P_{\Lambda^*}$ on the element $y = \operatorname{col}(f, \alpha)$ of the space $L_p^n \times R^m$ is equivalent to the action of the operator

$$\left[-P_{Q^*} l \int_a^b C(\cdot,\tau) * d\tau,\ P_{Q^*} *\right],$$

which coincides with the solvability criterion (4) of the Noetherian boundary value problem (1), (2).

THEOREM 2. The operator equation $\Lambda x = y$ is solvable for those and only those $y \in L_p^n \times R^m$ which satisfy the condition $P_{\Lambda^*} y = 0$ and, moreover, it has an r-parameter family of solutions of the form

$$x(t) = X_r(t) c_r + (\Lambda^- y)(t), \quad r = \dim\ker\Lambda,$$

where $\Lambda^-$ is the generalized inverse operator defined by formula (12).

2. We indicate another approach to the construction of the generalized inverse operator $\Lambda^-$ of a Noetherian operator $\Lambda: X \to Y$, acting from a Banach space X into a Banach space Y.

Let $\Lambda$ be a bounded linear Noetherian ($\dim\ker\Lambda = r$, $\dim\ker\Lambda^* = d$) operator, let $\Lambda^*$ be the adjoint operator of $\Lambda$, $\Lambda^*: Y^* \to X^*$, defined according to [12]

$$(\Lambda^* g)(x) = g(\Lambda x), \quad x \in X, \quad g \in Y^*,$$

where $X^*$ and $Y^*$ are the spaces conjugate to the spaces X and Y, respectively. As before, by $N(\Lambda)$, $N(\Lambda^*)$ and $R(\Lambda)$, $R(\Lambda^*)$ we shall denote the null-spaces (kernels) and the ranges of the operators $\Lambda$, $\Lambda^*$, respectively.

Let $\{f_i\}$, $i = 1,\dots,r$, be a basis of $\ker\Lambda$ and let $\{\varphi_s\}$, $s = 1,\dots,d$, be a basis of $\ker\Lambda^*$. As a consequence of the Hahn–Banach theorem [13], there exist linearly independent functionals $\gamma_j \in X^*$ such that

$$\gamma_j(f_i) = \delta_{ji}, \quad i, j = 1, \dots, r,$$

and also linearly independent elements $\psi_k \in Y$ such that

$$\varphi_s(\psi_k) = \delta_{sk}, \quad s, k = 1, \dots, d.$$

Following [3, 14], we define the projections

$$P_\Lambda: X \to N(\Lambda), \quad P_\Lambda^2 = P_\Lambda, \quad P_{\Lambda^*}: Y \to N(\Lambda^*), \quad P_{\Lambda^*}^2 = P_{\Lambda^*}$$

according to the formulas

$$P_\Lambda x = \sum_{i=1}^r \gamma_i(x) f_i, \quad P_{\Lambda^*} y = \sum_{s=1}^d \varphi_s(y) \psi_s.$$

Let $p = \min(r, d)$. Then, following [15], we introduce the operators

$$\overline{P}_\Lambda x = \sum_{i=1}^p \gamma_i(x)\psi_i, \quad \overline{P}_\Lambda: X \to \begin{cases} N_1(\Lambda^*) \subseteq N(\Lambda^*), & \text{if } r \leqslant d, \\ N(\Lambda^*), & \text{if } r \geqslant d; \end{cases}$$

$$\overline{P}_{\Lambda^*} y = \sum_{s=1}^p \varphi_s(y) f_s, \quad \overline{P}_{\Lambda^*}: Y \to \begin{cases} N(\Lambda), & \text{if } r \leqslant d, \\ N_1(\Lambda) \subseteq N(\Lambda), & \text{if } r \geqslant d. \end{cases}$$

For the construction of the generalized inverse operator we prove the following lemmas.

LEMMA 1. The operators $P_\Lambda$, $P_{\Lambda^*}$, $\overline{P}_\Lambda$, $\overline{P}_{\Lambda^*}$ have the following properties:

1) $P_{\Lambda^*}\overline{P}_\Lambda = \overline{P}_\Lambda P_\Lambda = \overline{P}_\Lambda$,

2) $P_\Lambda \overline{P}_{\Lambda^*} = \overline{P}_{\Lambda^*} P_{\Lambda^*} = \overline{P}_{\Lambda^*}$,

3) $\overline{P}_\Lambda \overline{P}_{\Lambda^*} y = \sum_{s=1}^p \varphi_s(y)\psi_s$,

4) $\overline{P}_{\Lambda^*}\overline{P}_\Lambda x = \sum_{i=1}^p \gamma_i(x) f_i$.

We prove, for example, property 1). We have $P_{\Lambda^*}\overline{P}_\Lambda x = \sum_{s=1}^d \varphi_s\left(\sum_{i=1}^p \gamma_i(x)\psi_i\right)\psi_s = \sum_{s=1}^d \sum_{i=1}^p \gamma_i(x)\varphi_s \times (\psi_i)\psi_s = \sum_{i=1}^p \gamma_i(x)\psi_i = \overline{P}_\Lambda x$, since $\varphi_s(\psi_i) = \begin{cases} \delta_{si}, & \text{if } i, s = 1, \dots, p, \\ 0, & \text{if } s > p, \end{cases}$ $\overline{P}_\Lambda P_\Lambda x = \sum_{i=1}^p \gamma_i\left(\sum_{j=1}^r \gamma_j(x) f_j\right)\psi_i = \sum_{i=1}^p \sum_{j=1}^r \gamma_j(x) \times \gamma_i(f_j)\psi_i = \sum_{i=1}^p \gamma_i(x)\psi_i = \overline{P}_\Lambda x$ since $\gamma_i(f_j) = \begin{cases} \delta_{ij}, & \text{if } i, j = 1, \dots p, \\ 0, & \text{if } j > p. \end{cases}$ The remaining properties are verified in a similar manner.

LEMMA 2. The operator $\bar{\Lambda} = \Lambda + \bar{P}_\Lambda$ has a bounded inverse

$$\bar{\Lambda}_{l,r}^{-1} = \begin{cases} (\Lambda + \bar{P}_\Lambda)_l^{-1} \text{ — left, if } r \leqslant d, \\ (\Lambda + \bar{P}_\Lambda)_r^{-1} \text{ — right, if } r \geqslant d. \end{cases}$$

Proof. Let $r \leqslant d$. From Theorems 2.1 and 5.1 from [10] there follows that for the existence of a left inverse operator $\bar{\Lambda}_l^{-1}$ of the operator $\bar{\Lambda}$ it is necessary and sufficient that a) $\ker \bar{\Lambda} = \{0\}$; b) $\dim \ker \bar{\Lambda}^* < \infty$.

We show that $\ker \bar{\Lambda} = \{0\}$. We assume that there exists $x_0 \neq 0$, $x_0 \in X$ such that $(\Lambda + \bar{P}_\Lambda) \times x_0 = 0$, from where $\Lambda x_0 = -\sum_{i=1}^{p} \gamma_i(x_0)\psi_i$. Applying to both sides of the last equality the functional $\varphi_s$, $s = 1,\dots,d$, we obtain

$$0 = \varphi_s(\Lambda x_0) = -\sum_{i=1}^{p} \gamma_i(x_0)\varphi_s(\psi_i) = -\gamma_s(x_0).$$

Thus, $\gamma_s(x_0) = 0$, $s = 1,\dots,p = r$. Since the system of functionals $\gamma_s$, $s = 1,\dots,r$, is linearly independent, by virtue of a consequence of the Hahn–Banach theorem [13], we have $x_0 = 0$. The obtained contradiction proves that $\ker \bar{\Lambda} = \{0\}$.

We show that $\dim \ker \bar{\Lambda}^* = d - r < \infty$. For this we find $\bar{P}_\Lambda^*$. If $x \in X$, $g \in Y$, then

$$g(\bar{P}_\Lambda x) = g\Big(\sum_{i=1}^{p} \gamma_i(x)\psi_i\Big) = \sum_{i=1}^{p} \gamma_i(x) g(\psi_i) = \sum_{i=1}^{p} g(\psi_i)\gamma_i(x) = \Big(\sum_{i=1}^{p} g(\psi_i)\gamma_i\Big)(x),$$

from where $\bar{P}_\Lambda^* g = \sum_{i=1}^{p} g(\psi_i)\gamma_i$.

We find the general form of the functional $g \in Y^*$, satisfying the equation

$$(\Lambda + \bar{P}_\Lambda)^* g = 0,$$

from where

$$\Lambda^* g = -\sum_{i=1}^{p} g(\psi_i)\gamma_i. \tag{14}$$

Applying (14) to the element $f_k$, we obtain

$$0 = (\Lambda^* g) f_k = g(\Lambda f_k) = -\sum_{i=1}^{r} g(\psi_i)\gamma_i(f_k) = -g(\psi_k), \quad k = 1,\dots,r.$$

Consequently, (14) has the form $\Lambda^* g = 0$. From here $g = \sum_{j=1}^{d} c_j\varphi_j$, where $\varphi_j$ are basis vectors of the kernel $N(\Lambda^*)$.

But we have established that $g(\psi_k) = 0$, $k = 1,\dots,p = r$; therefore,

$$0 = g(\psi_k) = \sum_{j=1}^{d} c_j\varphi_j(\psi_k) = \sum_{j=1}^{p} c_j\varphi_j(\psi_k) + \sum_{j=p+1}^{d} c_j\varphi_j(\psi_k).$$

From the fact that $\varphi_j(\psi_k) = \begin{cases} \delta_{jk}, & \text{if } j,k = 1,\dots,p, \\ 0, & \text{if } j > p, \end{cases}$ there follows that $c_j = 0$ for $j = 1,\dots,p = r$ and $c_j$ are arbitrary numbers for $j = r + 1,\dots,d$ and, therefore,

$$g = \sum_{i=1}^{d-r} c_i\varphi_i \in N(\bar{\Lambda}^*), \quad \dim \ker \bar{\Lambda}^* = d - r < \infty.$$

Thus, we have proved for the case $r \leqslant d$ the existence of the left inverse operator $\bar{\Lambda}_l^{-1}$ of the operator $\bar{\Lambda}$.

Since the bounded linear operator $\Lambda$ is Noetherian and, consequently [15], normally solvable, it follows that $R(\bar{\Lambda})$ is closed in Y. The closedness and, therefore, also the boundedness of the operator $(\Lambda + \bar{P}_\Lambda)_l^{-1}$ follows from the closedness of $R(\Lambda)$ and the finite-dimensionality of $N_1(\Lambda^*) = R(\bar{P}_\Lambda)$.

In order to prove the existence of the right inverse operator $\bar{\Lambda}_r^{-1}$ it is necessary and sufficient [10] to show that a) $\ker \bar{\Lambda}^* = \{0\}$; b) $\dim \ker \bar{\Lambda} < \infty$. The proof is similar to that of the case $r \leqslant d$. If r = d = p, then $\bar{\Lambda}_{l,r}^{-1} = \Lambda^{-1}$ and the formulated lemma turns into Schmidt's known lemma [3, 14].

LEMMA 3. The operator $\bar{\Lambda}_{l,r}^{-1}$ has the following properties

$$P_{\Lambda}\bar{\Lambda}_{l,r}^{-1} = \bar{P}_{\Lambda^*}, \tag{15}$$

$$\Lambda\bar{\Lambda}_{l,r}^{-1} = I_Y - P_{\Lambda^*}, \tag{16}$$

$$\bar{\Lambda}_{l,r}^{-1}P_{\Lambda^*} = \bar{P}_{\Lambda^*}, \tag{17}$$

$$\bar{\Lambda}_{l,r}^{-1}\Lambda = I_X - P_{\Lambda}. \tag{18}$$

We shall carry out the proof of the lemma for the case $r \leqslant d$. Since $\bar{P}_{\Lambda^*}\Lambda = 0$ and $\bar{P}_{\Lambda^*} \times \bar{P}_{\Lambda} = P_{\Lambda}$, applying on the right the operator $\Lambda + \bar{P}_{\Lambda}$ to both sides of (15), we obtain the identity

$$P_{\Lambda} = \bar{P}_{\Lambda^*}(\Lambda + \bar{P}_{\Lambda}) = \bar{P}_{\Lambda^*}\Lambda + \bar{P}_{\Lambda^*}\bar{P}_{\Lambda} = P_{\Lambda^*},$$

proving property (15). Since, by virtue of Lemma 1, $P_{\Lambda^*}\bar{P}_{\Lambda} = \bar{P}_{\Lambda}$ and $P_{\Lambda^*}\Lambda = 0$, applying to both sides of (16) the operator $\Lambda + P_{\Lambda}$ on the right, we obtain an identity that proves property (16):

$$\Lambda = (I_Y - P_{\Lambda^*})(\Lambda + \bar{P}_{\Lambda}) = \Lambda + \bar{P}_{\Lambda} - P_{\Lambda^*}\Lambda - P_{\Lambda^*}\bar{P}_{\Lambda} = \Lambda + \bar{P}_{\Lambda} - \bar{P}_{\Lambda} = \Lambda.$$

Since $P_{\Lambda^*}^2 = P_{\Lambda^*}$ and $\Lambda\bar{P}_{\Lambda^*} = 0$, applying on the left the operator $\Lambda$ to both sides of (17) and making use of (16), we obtain the identity

$$0 = (I_Y - P_{\Lambda^*})P_{\Lambda^*} = \Lambda\bar{\Lambda}_{l,r}^{-1}P_{\Lambda^*} = \Lambda\bar{P}_{\Lambda^*} = 0,$$

which proves property (17). Applying the operator $\bar{\Lambda}_{\ell}^{-1}$ on the right to both sides of (18) and making use of the properties (15)-(17), we obtain the identity

$$\bar{\Lambda}_{l,r}^{-1}I_Y - \bar{\Lambda}_{l,r}^{-1}P_{\Lambda^*} = \bar{\Lambda}_{l,r}^{-1}\Lambda\bar{\Lambda}_{l,r}^{-1} = (I_X - P_{\Lambda})\bar{\Lambda}_{l,r}^{-1} = I_X\bar{\Lambda}_{l,r}^{-1} - P_{\Lambda}\bar{\Lambda}_{l,r}^{-1},$$

which proves property (18).

For the case $r \geqslant d$ the proof of the lemma is carried out in a similar manner.

The above given lemmas enable us to prove the following theorem [16].

THEOREM 3. The operator

$$\Lambda^- = \bar{\Lambda}_{l,r}^{-1} - \bar{P}_{\Lambda^*} \tag{19}$$

is a bounded generalized inverse of the bounded linear Noetherian operator $\Lambda$.

For the proof of the theorem we verify that $\Lambda^-$ satisfies the properties (13). For this, first we show that

$$\Lambda\Lambda^- = I_Y - P_{\Lambda^*}, \quad \Lambda^-\Lambda = I_X - P_{\Lambda}.$$

Indeed,

$$\Lambda\Lambda^- = \Lambda(\bar{\Lambda}_{l,r}^{-1} - \bar{P}_{\Lambda^*}) = \Lambda\bar{\Lambda}_{l,r}^{-1} - \Lambda\bar{P}_{\Lambda^*} = I_Y - P_{\Lambda^*},$$

since

$$\Lambda(\bar{P}_{\Lambda^*}y) = \Lambda\Big(\sum_{s=1}^{p}\varphi_s(y)f_s\Big) = \sum_{s=1}^{p}\varphi_s(y)\Lambda f_s = 0;$$

$$\Lambda^-\Lambda = (\bar{\Lambda}_{l,r}^{-1} - \bar{P}_{\Lambda^*})\Lambda = \bar{\Lambda}_{l,r}^{-1}\Lambda - \bar{P}_{\Lambda^*}\Lambda = I_X - P_{\Lambda},$$

since $\bar{P}_{\Lambda^*}(\Lambda x) = \sum_{s=1}^{p}\varphi_s(\Lambda x)f_s = \sum_{s=1}^{p}(\Lambda^*\varphi_s)(x)f_s = 0$. Now we verify property (13). We have

$$\Lambda\Lambda^-\Lambda = \Lambda(I_X - P_{\Lambda}) = \Lambda - \Lambda P_{\Lambda} = \Lambda,$$

$$\Lambda^-\Lambda\Lambda^- = (I_X - P_{\Lambda})\Lambda^- = \Lambda^- - P_{\Lambda}\Lambda^- = \Lambda^-,$$

since $P_{\Lambda}\Lambda^- = P_{\Lambda}\bar{\Lambda}_{l,r}^{-1} - P_{\Lambda}\bar{P}_{\Lambda^*} = \bar{P}_{\Lambda^*} - \bar{P}_{\Lambda^*} = 0$ by virtue of Lemmas 1 and 3. Thus, the theorem is proved.

3. The method, presented in Sec. 1, of the construction of the generalized inverse operator $\Lambda^-$ of the boundary value problem (1), (2) with the aid of the generalized Green

operator is valid also in the case when the operator L is bounded linear Noetherian [without the assumption of the solvability for an arbitrary $f \in L_p^n$ of the Cauchy problem for (1)].

Indeed, let $\dim\ker L = s$ and let $P_{L^*}: L_p^n \to \ker L^*$ be the projection whose construction is described in Sec. 2. Then Eq. (1) is solvable for those and only those $f \in L_p^n$ for which

$$P_{L^*} f = 0, \tag{20}$$

and we have an s-parameter family of solutions

$$x(t) = X_s(t) c_s + (L^- f)(t), \quad c_s \in R^s,$$

where $X_s(t)$ is an $n \times s$ matrix, whose columns form a basis of $\ker L$, while $L^-$ is the generalized inverse operator (19).

The boundary value problem (1), (2) with a Noetherian operator $L: D_p^n \to L_p^n$ is solvable if and only if $f \in L_p^n$ and $\alpha \in R^m$ satisfy the conditions (20) and

$$P_{Q_{d_1}^*} \{\alpha - l(L^- f)\} = 0 \qquad (d_1 = m - \operatorname{rank} Q_1)$$

and, moreover, it has an $r_1$-parameter family of solutions ($r_1 = n - \operatorname{rank} Q_1$)

$$x(t) = X_{r_1}(t) c_{r_1} + X_s(t) Q_1^+ \alpha + (G_1 f)(t),$$

where $X_{r_1}(t) = X_s(t) P_{Q_{r_1}}$ is the $n \times r_1$ fundamental matrix of the boundary value problem (1), (2), $Q_1^+$ is an $s \times m$ constant matrix, the generalized inverse of the $m \times s$ matrix $Q_1 = \ell X_s$, while $G_1$ is the generalized Green operator of the semihomogeneous boundary value problem (1), (2), having the representation

$$(G_1 f)(t) = (L^- f)(t) - X_s(t) Q_1^+ l(L^- f).$$

Moreover, the generalized inverse operator $\Lambda^-$, resolving the boundary value problem (1), (2) with a Noetherian operator L, has the form

$$\Lambda^- * = [G_1 *, X_s Q_1^+ *].$$

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